AD-A109 657 STANFORD UNIV CA DEPT OF STATISTICS F/G 12/1
GROWTH OF THE MAXIMUM IN A CRITICAL AGE-DEPENDENT BRANCHING PRO--ETC(U)
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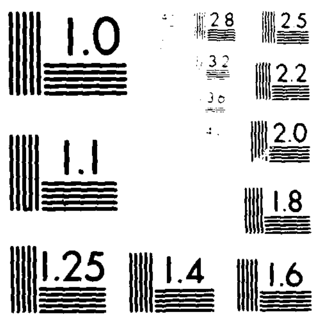

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GROWTH OF THE MAXIMUM IN A CRITICAL AGE-DEPENDENT BRANCHING PROCESS

By

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TECHNICAL REPORT NO. 306

SEPTEMBER 15, 1981

Prepared Under Contract
N00014-76-C-0475 (NR-042-267)
For the Office of Naval Research

Herbert Solomon, Project Director

DEPARTMENT OF STATISTICS
STANFORD UNIVERSITY
STANFORD, CALIFORNIA

Growth of the maximum in a critical age-dependent branching process

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I. Introduction.

Let $Z(t)$ denote the number of cells alive at time $t \geq 0$ in a critical age-dependent branching process with offspring generating function $h(s)$ and absolutely continuous cell lifetime distribution $G(t)$. It is assumed that $h(s)$ and $G(t)$ have finite second moments. The process evolves by starting with one newborn cell at time $t = 0$. At the end of its life, the distribution of offspring cells follows $h(s)$. Each new cell proceeds independently and identically as every other cell and independent and identically of the parent cell.

Let

$$M(t) \equiv \max_{0 \leq s \leq t} Z(t). \tag{1.1}$$

It is shown by a comparison method that $EM(t) \sim c(t) \log t$, and $\operatorname{Var} M(t) \sim b(t)t$, $0 < d < c(t)$, $b(t) < c < \infty$ for t sufficiently large.

II. Galton-Watson process.

Let $\{Z_n\}$, $n \geq 0$, with $Z_0 \equiv 1$ denote the number of cells in a critical Galton-Watson branching process in discrete time with (Athreya, Ney (1970), pp. 6-7)

$$k_n(s) \equiv Es^{Z_n} = \frac{np - (np-q)s}{np + q - nps} \tag{2.1}$$

where $p > 0$, $q > 0$, $p + q = 1$ and are to be chosen in Section III.

Let

$$M_n = \max_{1 \leq \ell \leq n} Z_\ell . \tag{2.2}$$

Lemma 1. For $1 \ll r \leq n$

$$\frac{b}{r} \leq P[M_n \geq r] \leq \frac{1}{r} \tag{2.3}$$

for some $0 < b \leq 1$.

For $r \geq n$,

$$P[M_n > r] \leq (1 + \frac{q}{np})^{-r} \tag{2.4}$$

Proof. Since $\{Z_n\}$ is a non-negative martingale with $EZ_n = 1$, the right side of (2.3) follows by Kolmogorov's maximal inequality.

The left side of (2.3) follows from

$$P[M_n \geq r] \geq P[Z_r \geq r] = \frac{q}{rp+q}(1 + \frac{q}{rp})^{r-1} \tag{2.5}$$

so that

$$P[M_n \geq r] \geq \frac{q}{rp+q}\, e^{-q/p} \simeq \frac{b}{r},$$

where

$$b = \frac{q}{p}\, e^{-q/p}. \tag{2.6}$$

To prove (2.4), let

$$T = \begin{cases} \min\{k,\ 1 \leq k \leq n \text{ such that } Z_k > r\} \\ n \text{ if } Z_k \leq r, \quad \text{all } 1 \leq k \leq n \end{cases}$$

Then

$$E[Z_T; Z_T > r] \geq rP[Z_T > r] = rP[M_n > r], \tag{2.7}$$

and repeated use of the martingale property,

$$E[Z_T; Z_T > r] = E[Z_n; Z_T > r] \leq E[Z_n; \bigcup_{\ell=1}^{n} \{Z_\ell > r\}]$$

so that

$$E[Z_T; Z_T > r] \leq \sum_{\ell=1}^{n} E[Z_\ell; Z_\ell > r]. \tag{2.8}$$

Hence (2.7), (2.8) yield

$$P[M_n > r] \leq \frac{1}{r} \sum_{\ell=1}^{n} E[Z_\ell; Z_\ell > r]. \tag{2.9}$$

By a computation using (2.1),

$$E[Z_\ell; Z_\ell > r] = \left(\frac{\ell p}{\ell p+q}\right)^{r-1}\left(\frac{rq+\ell p}{\ell p+q}\right) \tag{2.10}$$

Then (2.9), (2.10) suffice for (2.4).

<u>Lemma 2</u>. Under the hypotheses of Lemma 1,

$$EM_n \sim a_n \log n, \; 0 < d < a_n < c \tag{2.11}$$

$$EM_n^2 \sim b_n n, \qquad 0 < d < b_n < c \tag{2.12}$$

for n sufficiently large, and c, d denote positive finite constants.

<u>Proof</u>.

$$\sum_{r=1}^{n} P[M_n \geq r] \leq EM_n = \sum_{r=1}^{n} P[M_n \geq r] + \sum_{r=n+1}^{\infty} P[M_n \geq r]. \tag{2.13}$$

By (2.3), for $1 \leq r \leq n$, there exist positive constants a, d such that

$$a \leq rP[M_n \geq r] \leq d. \tag{2.14}$$

By (2.4), the second sum on the right of (2.13) is a convergent series. Then (2.14) applied to the other sum on both sides of (2.13) suffices for (2.11).

The expression (2.12) follows by the same argument using the expression

$$EM_n^2 \simeq 2 \sum_{r=1}^{\infty} rP[M_n \geq r]. \tag{2.15}$$

III. Age-Dependent case.

Theorem. Let a critical age-dependent branching process with absolutely continuous lifetime distribution function G(t) and offspring generating function h(s), both have finite second moments, then

$$EM(t) = a(t)\log t \tag{3.1}$$

$$\mathrm{Var}\, M(t) = b(t)t \tag{3.2}$$

where $0 < d < a(t) < c < \infty$, $0 < d < b(t) < c < \infty$, for c, d constants.

Proof. Since G(t) is absolutely continuous, the split times of the Z(t) process are distinct a.s.

Let $\{W_n\}$, $n \geq 1$ denote a critical Galton-Watson process with given offspring generating function h. By Spitzer's comparison lemma (Athreya and Ney, 1970, p. 22) and its extension to joint distributions (see, e.g. Weiner (1978), pp. 216-217), there exist critical Galton-Watson processes $\{Z_{0n}\}$, $\{Z_n\}$ each with fractional-linear offspring generating function with positive variances, an $0 < s_0 < 1$, and an $M > 0$ so that for $s_0 < s_\ell < 1$, $1 \leq \ell \leq m$, and $M < n_1 < n_2 < \cdots < n_m$,

$$E\left[\prod_{\ell=1}^{m}\left(s_\ell^{Z_{0n_\ell}}\right)\right] \leq E\left[\prod_{\ell=1}^{m}\left(s_\ell^{W_{n_\ell}}\right)\right] \leq E\left[\prod_{\ell=1}^{m}\left(s_\ell^{Z_{n_\ell}}\right)\right]. \tag{3.3}$$

From the form of the terms $P[Z_n = j]$ where $\{Z_n\}$ is a critical Galton-Watson process with fractional linear offspring generating function, one may conclude that

$$P\left[\bigcup_{\ell=1}^{m} Z_\ell > j\right] \text{ is increasing in } 0 < \sigma^2 \equiv E(Z_1-1)^2 < \infty. \tag{3.4}$$

From (3.3), (3.4) it follows that there are critical Galton-Watson processes with fractional linear offspring generating functions with positive variances $\{Z_{in}\}$, $1 \leq i \leq 4$, such that for all n sufficiently large, with

$$\begin{aligned} M_n &= \max_{1\leq \ell\leq n} W_\ell \\ M_{in} &= \max_{1\leq \ell\leq n} Z_{i\ell}, \qquad 1 \leq i \leq 4 \end{aligned} \tag{3.5}$$

such that

$$\begin{aligned} EM_{1n} \leq EM_n \leq EM_{2n} \\ \operatorname{Var} M_{3n} \leq \operatorname{Var} M_n \leq \operatorname{Var} M_{4n}. \end{aligned} \tag{3.6}$$

From (1.3), (1.4), (2.4) of (Esty (1975) pp. 49-50), an induction yields that for $0 < \alpha_1 < \alpha_2 < \cdots < \alpha_m$, $\int_0^\infty t dG(t) = \mu > 0$, $0 < s_\ell < 1$, $1 \leq \ell \leq m$, and $n_\ell = [\alpha_\ell t/\mu]$, $t_\ell = \alpha_\ell t$, $1 \leq \ell \leq m$, that

$$\lim_{t\to\infty} t\left|E\left[\prod_{\ell=1}^{m}\left(s_\ell^{Z(t_\ell)}\right)\right] - E\prod_{\ell=1}^{m}\left(s_\ell^{W_{n_\ell}}\right)\right| = 0. \tag{3.7}$$

It follows that for t sufficiently large, and $n = [t/\mu]$, that

$$\begin{aligned} |EM_n - EM(t)| \to 0 \\ |\operatorname{Var} M_n - \operatorname{Var} M(t)| \to 0 \end{aligned} \tag{3.8}$$

so that (3.6), (3.8) imply that for $n = [t/\mu]$, and $t \to \infty$, that

$$
\begin{aligned}
EM_{1n} &\le EM(t) \le EM_{2n} \\
\operatorname{Var} M_{3n} &\le \operatorname{Var} M(t) \le \operatorname{Var} M_{4n}.
\end{aligned}
\tag{3.9}
$$

Then (2.11), (2.12) and (3.9) yield (3.1), (3.2), upon replacing n by $[t/\mu]$. This completes the theorem.

IV. Remarks.

The distribution of the absolute maximum of a critical Galton-Watson process over all time until extinction and the application of this result to critical age-dependent processes has been obtained in (Lindvall (1976)) by different methods.

This approach, combining easily estimable quantities from the critical Galton-Watson process with fractional-linear offspring generating function with the asymptotic approximations in (Esty (1975), pp. 49-50) may be used to obtain asymptotic results for critical age-dependent branching processes. For example, if T = time to extinction, then $tP[T > t] \to \alpha > 0$.

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SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER 306	2. GOVT ACCESSION NO. AD-A109657	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) GROWTH OF THE MAXIMUM IN A CRITICAL AGE-DEPENDENT BRANCHING PROCESS		5. TYPE OF REPORT & PERIOD COVERED TECHNICAL REPORT
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) HOWARD WEINER		8. CONTRACT OR GRANT NUMBER(s) N00014-76-C-0475
9. PERFORMING ORGANIZATION NAME AND ADDRESS Department of Statistics Stanford University Stanford, CA 94305		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS NR-042-267
11. CONTROLLING OFFICE NAME AND ADDRESS Office Of Naval Research Statistics & Probability Program Code 436 Arlington, VA 22217		12. REPORT DATE SEPTEMBER 15, 1981
		13. NUMBER OF PAGES 8
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)		15. SECURITY CLASS. (of this report) UNCLASSIFIED
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

APPROVED FOR PUBLIC RELEASE: DISTRIBUTION UNLIMITED.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Logarithm; growth; maximum; critical branching process.

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

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GROWTH OF THE MAXIMUM IN A CRITICAL

AGE-DEPENDENT BRANCHING PROCESS

Let $Z(t)$ denote the number of cells alive at time t in a critical age-dependent branching process starting with one new cell at $t = 0$ and let $M(t) = \max_{0 \leq s \leq t} Z(s)$. Under suitable moment assumptions and an absolutely continuous lifetime distribution function it is shown that $EM(t) \sim c(t) \log t$, $\operatorname{Var} M(t) \sim b(t)t$, $0 < d < b(t), c(t) < c < \infty$ for t sufficiently large. The method is by comparison with critical fractional-linear Galton-Watson processes.

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